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A praxeological approach to Klein's plan B: cross-cutting from Calculus to Fourier Analysis

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In our previous work on Calculus–Analysis transition we independently explored the reasons of students' difficulties with studying analysis and observed that the problem is related to the discontinuity of students' experiences leading to their inability to interpret the (formal and more rigorous) ideas learned in analysis courses in terms of (practical) knowledge acquired in calculus courses, and vice versa. In this paper we continue and combine our work with two new contributions: a theoretical formulation of Klein's idea of a "Plan B" for teaching mathematics, applied to the transition in question; and a concrete student activity attempting to give flesh to this "plan" for the special case of introductory Fourier Analysis.

Keywords: Calculus, Fourier analysis, transition, praxeology

Introduction

Calculus and Analysis appear as related, but distinct subdisciplines in many contemporary university programmes. Calculus courses specialise in mathematical themes indicated by course titles such as "Integral Calculus", "Functions of several variables" or "Ordinary differential equations". Analysis courses, on the other hand, treat theoretical perspectives on these same mathematical themes, gradually moving from course titles such as "Real Analysis", "Fourier Analysis" towards more abstract areas such as functional and harmonic analysis. In short, calculus courses can be roughly characterized as teaching students certain calculation practices related to real and vector valued functions, with little theoretical precision or justification – while analysis courses tend to present "formal theory with little practice". This division is a didactical construct which is related to historical and institutional conditions (see Klisinska, 2009, for an in-depth analysis of the case of the "fundamental theorem of calculus").

The main reason for the division seems to be that the two types of courses cater to different student populations. While calculus courses are studied by a large cohort of students in natural and social sciences, much fewer students get to study analysis (mainly students of pure mathematics, theoretical physics and mathematical statistics). For these and other reasons, it may be difficult to change the course structure.

The transition from Calculus to Analysis presents mathematics students with several challenges (for examples, see Winsløw & Grønbæk, 2014). Here is a typical student formulation of some of these (interview with a student of the first author, summer 2016):

> In calculus courses we learn methods, but usually the why questions are not explained or proved. (...) However, analysis courses felt as separate. They were more theoretical than applied. I never grasped them as well as Calculus. It was often unclear, what it was leading to. I wish we had a

better sense of connection between the theory we covered in pure math courses and the methods shown in applied math courses.

We have explored this perceived lack of "connection" in earlier papers (Kondratieva, 2011, 2015; Winsløw, 2007, 2016). In the present paper, we use the notion of praxeology (Chevallard, 2006) to represent the general "connection" problem in more precise terms, and - as a theoretical case study - to present a new proposal for "connecting" Calculus and one of the basic theorems in Fourier Analysis. Our research results are thus basically theoretical.

Theoretical framework

Chevallard (2006) defines a *praxeology* as a pair (P, L) consisting of a *praxis block P* and a *logos block L*. A praxeology is a minimal element of human knowledge, *P* representing the practical part - the "know how" - and *L* the intellectual part, the "thinking and explaining" – often called "know why". The two are interdependent:

> …no human action can exist without being, at least partially, "explained", made "intelligible", "justified", "accounted for", in whatever style of "reasoning" such an explanation or justification may be cast. Praxis thus entails logos, which in turn backs up praxis. For praxis needs support – just because, in the long run, no human doing goes unquestioned. (Chevallard, 2006, p. 23).

As we focus here on mathematical praxeologies taught and learnt at university, it is obvious that *praxis* (e.g. computing the Fourier series of a given function) is intimately connected to various forms of *logos* - from *ad hoc* explanations of standard techniques to theories involving general definitions, theorems and proofs. To compare and contrast the praxeologies developed in calculus and analysis courses, we consider that they represent various affinities with the praxeologies of present-day mathematicians, which we shall represent suggestively using Greek letters (Π, Λ). We can thus, as a first naïve model, propose that praxeologies taught and learnt in calculus courses are of the form (Π_i, L_i): the *praxis blocks,* including computational techniques, are identical to those used (for tasks of the same type) by professional mathematicians, while the logos blocks L_i are limited to informal explanations of smaller collection of practice blocks (like the various techniques for determining whether a series is convergent or not). On the other hand, analysis courses then focus on the scientific form of logos blocks. The taught and learnt praxeologies in such courses are therefore of the form (P_i, Λ_i) where each Λ_i constitutes a logos block consistent with the scientific model, while the praxis blocks P_i are didactic "afterthoughts" constructed to consolidate the acquisition of Λ_i. As mentioned in the introduction – such teaching practices often fail to motivate students for Λ_i and to provide them with a coherent, autonomous relationship with (Π_i, Λ_i). Our research focuses on how this issue can be addressed.

Taken together, calculus and analysis courses in principle provide students with praxeologies (Π_i, Λ_i) which, taken individually, are close to the scientific model. For instance, convergence tests used in Calculus praxis on series are now supplied with a theory involving precise definitions and proofs of the "criteria" for convergence. However, because the number and technical complexity of these praxeologies is quite high and the Π_i were taught in other courses, typically years before, some effort and support may still be needed for students to "assemble" individual praxeologies (Π_i, Λ_i). We can say that working along these lines corresponds to establishing complete but separate praxeologies within different small areas of mathematics, which is what Klein called "Plan A" for teaching: "Plan A is based upon more particularistic conception of science which divides the total field into a series of mutually separated parts and attempts to develop each part by itself." (Klein 1908/1932, p. 77, see also Winsløw, 2016) Within this approach two praxeologies are related only through strict logical dependency at the theoretical level and only within strictly confined areas (which, in terms of what students actually acquire, may be surprisingly small).

However as explained by Klein, the scientific practice (historically as well as currently) involves more than isolated or strictly dependent praxeologies. Klein (1908/1932, p. 78) recommended that also elements of "Plan B" be included in mathematics teaching both in schools and at university:

> The supporter of Plan B lays the chief stress upon the organic combination of the partial fields, and upon the stimulation which these exert one upon another. He prefers, therefore, the methods which open for him an understanding of several fields under a uniform point of view.

In terms of the praxeological model above, we may thus summarize the two "plans" or strategies for developing and connecting students' previous knowledge as follows:

Plan A. *assemble elementary praxeologies* (Π_i, Λ_i) from calculus and analysis elements, by establishing firm relations of type $\Pi_i \leftrightarrow \Lambda_i$. In fact, this is sometimes a possible function of the "fingertip" exercises, which constitute P_i in many courses and textbooks on analysis.

Plan B. *develop cross-cutting relationships among praxeologies* which could be of one of the following types (or combinations among them):

B1. Relating praxis blocks $(\Pi_i \leftrightarrow \Pi_j)$ or logos blocks $(\Lambda_i \leftrightarrow \Lambda_j)$

B2. Relating otherwise unrelated praxis and logos blocks $(\Pi_i \leftrightarrow \Lambda_j)$

It may be more easy and common to develop relations of type B1, even if they certainly appear more often in "mathematician" praxeologies than in typical course teaching. We now present and analyse an example of student activity aiming at developing relations of the last type (B2): namely, that students connect a collection of praxis blocks Π_i (concerning trigonometry, integration and convergence) to a logos block (Λ_0) from Fourier Analysis.

A logos block from Fourier Analysis

For a 2π-periodic, piecewise continuous function $f: \mathbb{R} \to \mathbb{C}$, the Fourier series of f is defined as

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ , where } a_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \text{ and } b_n = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx .$$

In general the two infinite series may not converge at a point *x*. In 1829, Dirichlet gave one of the first sufficient conditions for pointwise convergence of a Fourier series. A version of this result which is usually formulated for *piecewise* continuous functions, is stated below in a special case to avoid technicalities. We refer to it as Dirichlet's theorem, although we don't use his original claim.

Theorem If $f: \mathbb{R} \to \mathbb{C}$ is a continuous 2π-periodic function with piecewise continuous derivative, the Fourier series of f is pointwise convergent to $f(x)$ at every $x \in \mathbb{R}$.

It will be important for the sequel to outline the main steps of the proof; it actually appears (often for the more general case mentioned) in any formal course on Fourier Analysis. The proof is based on the following essential steps (see e.g. Folland, 1992, pp. 30–36 for the wealth of computational details omitted here):

1. First, it is shown that under weaker assumptions, such as f being square integrable and 2π-periodic, the coefficients a_n and b_n tend to zero as n tends to infinity. (In fact, one demonstrates this by showing that the series $\sum a_n^2$ and $\sum b_n^2$ are both convergent.)
2. Next, by direct computation, we rewrite the Nth partial sum given by $s_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos nx + \sum_{n=1}^{N} b_n \sin nx$ as: $s_N(x) = \int_{-\pi}^{\pi} f(x+y) K_N(y)\,dy$, where K_N is the *Dirichlet kernel* defined by $K_N(x) = \frac{1}{\pi}(\frac{1}{2} + \sum_{n=1}^{N} \cos nx)$. Clearly $K_N(0) = (N+1/2)/\pi$ and $\int_{-\pi}^{\pi} K_N(x)\,dx = 1$. By clever use of the addition formulae, $K_N(x) = \dfrac{\sin(Nx + x/2)}{2\pi \sin(x/2)}$ for $x \neq 0$.
3. Finally, using 2., a straightforward set of computations yields

$$(*) \qquad s_N(x) - f(x) = \tfrac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\sin(y/2)\cos Ny\,dy + \tfrac{1}{\pi}\int_{-\pi}^{\pi} g_x(y)\cos(y/2)\sin Ny\,dy,$$

   where g_x is defied by $g_x(y) = \dfrac{f(x+y) - f(x)}{2\sin(y/2)}$ for $y \neq 0$, and $g_x(0) = f'(x)$. In fact, $g_x(y)$ is a continuous for $y \neq 0$ and 2π-periodic function. At $y = 0$ $g_x(y)$ may have a jump discontinuity if $f'(x+0) \neq f'(x-0)$. The functions $g_x(y)\sin(y/2)$ and $g_x(y)\cos(y/2)$, notwithstanding the possible discontinuity of the latter at $y = 0$, are bounded, and thus, square integrable. Formula (*) shows that $s_N(x) - f(x)$ is simply the sum of Nth Fourier coefficients of these two functions. Applying now 1., we get the desired result.

The key point of the proof is (*): to rewrite the "tail" of the Fourier series as a sum of two Fourier coefficients, together with the fact that the coefficients tend to zero. While, according to the distinction we made above, the general result (and certainly its proof) does not belong to the realm of Calculus, most of the techniques are, locally, well known from calculus courses. When students are presented with the theory - as mentioned, in a somewhat more general form, - they may not realize that the proof almost entirely draws on well-known techniques. To make them discover that is the aim of the design that we present in the next section, focusing on the following special case:

Example. Applying the above Theorem for $f(x) = x^2$, extended periodically from $[-\pi, \pi]$ to $\mathbb{R}$, we get that the Fourier series converges to 0 at $x = 0$. Computing the Fourier coefficients, this gives

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \text{so that} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

The latter - striking - result can be derived by other means, as a variant of the famous Basel problem (see e.g. Kondratieva, 2016). One such approach is at the root of the design presented below.

Outline and *a priori* analysis of Student Activity

In continuation of earlier work of the first author (Kondratieva, 2016), we took the Example above as a point of departure for constructing a sequence of exercise-like activities that would lead students through two approaches to computing the infinite sum considered in the Example: *part 1* consisting of a series of "calculus-like assignments" which, without saying so, go through the proof of Dirichlet's theorem in the special case where $f(x) = x^2$; *part 2* in which the students work directly with the result, as in the Example; and a *final reflection* in which the students are supposed to realize that the proof (known from a Fourier Analysis logos block Λ_0) amounts to nothing more than a generalization of the sequence of calculus techniques drawn upon in part 1. We notice here that the numbering suggests that the praxis and logos blocks thus connected through the activity are not, *prima facie*, connected - and, thus, the connection established is really of type B.

Part 1 begins with presenting the problem of determining the value of $S = \sum (-1)^{n+1}/n^2$. The praxis blocks acquired in calculus courses do not provide ready-made techniques to solve this problem; instead, students are invited to do so through "several preliminary problems":

1. Compute the integral $\int_{-\pi}^{\pi} x^2 \cos mx \, dx$ for any natural number m (Π_1 : integration rules).
2. Show that $\frac{1}{2} + \sum_{n=1}^{m} \cos nx = \dfrac{\sin(m+1/2)x}{2\sin(x/2)}$ (Π_2 : trigonometric formulae).
3. Show that $u(x) = \begin{cases} x^2/\sin x, & x \neq 0 \\ 0 & x = 0 \end{cases}$ defines a continuous function on $[0, \pi/2]$

   (Π_3 : techniques to compute limits, including the special result $\lim_{x\to 0} x^{-1}\sin x = 1$).
4. Find u' and show this function is bounded on $[0, \pi/2]$

   (Π_3, and Π_4 : differentiation from first principles).
5. Show $\int_{-\pi}^{\pi} x^2 \dfrac{\sin(m+1/2)x}{2\sin(x/2)} dx = 8\int_0^{\pi/2} y^2 \dfrac{\sin(2m+1)y}{\sin y} dy = 8\int_0^{\pi/2} \dfrac{\cos(2m+1)y}{2m+1} u'(y) dy$ (Π_1).
6. Show that the integral in 5 converges to 0 as $m \to \infty$ (Π_1 and Π_3).
7. Finally, combine the results above to find S (Π_1).

The only slightly advanced praxis (technique) involved in the above appears in 6., where Π_1 is supposed to include something like $\left|\int f\right| \leq \int |f|$ – or, alternatively, Π_3 should include a rule which permits to conclude that $\lim_{m\to\infty} \int_I f_m = 0$ under appropriate conditions on (f_m).

Part 2 of the activity invites the students to compute the Fourier series of $f(x) = x^2$ and engage in some concrete computations related to its convergence which are in fact very similar to 1.-6. above.

This prepares the final part, which, even if the students have forgotten or never seen the general proof of Dirichlet's theorem, is supposed to make them discover the close parallel between the two parts.

In case they do recall elements of the proof above, they will recognize the essential ideas of steps 2 and 3. Meanwhile, step 1 appears more indirectly in the concrete case, where both the Fourier coefficients of f and the auxiliary coefficients, appearing in (*), can be computed or estimated directly. Indeed, many textbooks present Step 1 as a corollary of a more general theorem on orthogonal sets in Hilbert space. This, together with the technicalities related to the possible non-continuity of f, contributes to the impression that the proof is way beyond simple techniques from Calculus. Nevertheless, comparing the proof with the proposed activity, the students could realize that it relies in essence on well known praxis blocks (Π_i, i = 1,2,3,4), only substituting a concrete function with the general function f. Certainly, this could establish a strong relation $\Lambda_0 \leftrightarrow$ (Π_i, i = 1–4) which might in fact be prepared by students' working *Part 1* above *prior* to encountering Λ_0.

Some experimental observations

To pilot and refine the above design before testing it with a larger group of students in a course on Fourier Analysis, we have done preliminary observations and interviews with five students who have completed at least 3 years of undergraduate mathematics program. These students were involved in summer research projects in mathematics at the Memorial University of Newfoundland. This involvement is an indicator of the students' high motivation and achievements in studying mathematics. The students were asked to solve the problems from the activity described above and participate in a follow-up semi-structured interview. The students were asked whether they found the problems (a) familiar, (b) interesting, (c) easy/accessible; and whether they saw any connections between praxis and logos of parts 1 and 2. All students regarded problems 1-6 as familiar from their calculus courses, and they found them easy. In words of one student, "I loved that stuff when I was in my calculus courses, so I found these problems pleasant… And they are not difficult, too." While problems 1-6 were familiar to the students, they clearly indicated that no projects of nature similar to problem 7 were present in their study: "I think it is a cool layout. Nothing of this format was in my calculus courses, – when you need to use previous results to solve larger or more interesting problem." Students regarded the task of series evaluation as challenging but also most enjoyable: "The problems 1-6 were like baby steps… And they met together nicely in problem 7". So, at least these students were successful and appreciative of tasks in part 1. As for the accessibility of part 1 for an average student in a calculus course, we had overall a confirmative response: "I think it is accessible for a student who has done Integral Calculus.... if they are not confined to a very short period of time, then yes." Another student confirmed, "it could be a good exam sequence, more fun than just doing problems." However, a different perspective was also articulated: "…many students take this [Integral Calculus] course because it is a prerequisite for their programs, so maybe they would not be interested as much."

Among the five students only one had studied Fourier series in his courses, while others had heard the term but had very little familiarity with the subject. However, they all recognized the similarity in the technical praxis of parts 1 and 2, for example, that calculation of the Fourier series in part 2 resembles evaluation of integrals in problem 1 from part 1. Bridging the theory and connecting the idea of convergence of an individual series in part 1 and pointwise convergence of Fourier series was more challenging. This is where the role of an instructor might be critical: to help students to relate new theoretical constructs and ideas to familiar praxis. We realize that students' background makes a difference, however even learners previously unfamiliar with Fourier series seem to benefit from this activity. Students' responses based not on reproduction of known facts, but rather on reasoning related to their practical experiences, is an indication of establishing new mathematical relations. The following is an excerpt from an interview with students of the first author:

M.K.: Is it always possible to replace a function with its Fourier series in calculations?

Student 1: In my (applied) courses we were told that no (a function is not always equal to its Fourier series), but this was never proved. Now it kind of makes more sense.

M.K.: Do you think that familiarity with part 1 would help to exemplify general theory related to Fourier series and their convergence?

Student 1: Yes, definitely. I think it is more logical to go this way about discussing conditions of pointwise convergence of Fourier series. However, the experiences need to be close together in time, so that the second part occurs before students have forgotten the first portion.

The space available does not allow us to give the details of students' accomplishments and their impact on our design. We simply note that the sample students were by and large able to complete them and see the inner connections. Also, the students considered that building on the familiar computational tasks (1-6) on the one hand, and on new theoretical constructs (Example) on the other hand, organized around given problem (evaluation of the series S) was stimulating: "Suppose someone has a theoretical solution and I have a computational solution and they look completely different, but they give the same answer to the same problem so they have to be the same somehow… then I want to go back and find out why they are the same. I found it very interesting."

Conclusions

While calculus courses include praxis blocks Π_i compatible with those of professional mathematicians, their theoretical components are more informal and focused on algebraic computation. Moreover, these praxis blocks are often isolated from each other, as they occur within separate sections of textbooks and courses, and students typically don't get opportunities to apply them in combinations. When students meet Dirichlet's theorem, they are given a general and relatively complicated proof (in Analysis). In such courses, "simple applications" (such as the Example above) may be introduced as examples or exercises, to build an artificial practice block P_0 corresponding to the much richer logos block Λ_0. The fact that the general proof (Λ_0) is essentially linked to familiar praxis blocks from Calculus will then not appear. We propose that by replacing P_0 by a sequence of computational auxiliary tasks (1-7), similar to the steps 2 and 3 of the proof (Λ_0), two goals can be achieved. First, students will see how different praxis blocks $(\Pi_1,...,\Pi_4)$ from Calculus work together and combine to support P_0 by themselves. Secondly, this special case might help to prepare for the various general theorems on Fourier series convergence (Λ_0 and beyond) by relating it to the concrete and familiar elements $\Pi_1,...,\Pi_4$. This hypothesis will be investigated empirically. More generally, we hypothesize that situations which enable students to establish "cross cutting relations" $\Pi_i \leftrightarrow \Lambda_j$ are precise and possibly partial interpretations of Klein's Plan B. At the same time, constructing integrated praxis blocks such as $(\Pi_1,...,\Pi_4)$ above constitutes an essential complement to "Plan A" type courses. Their construction will clearly necessitate a careful analysis of (central) theory blocks of more advanced courses, and resources found in reasonably well-established praxis blocks of previous courses. So, while the general hypothesis may look fairly simply, realizing it in concrete cases - even theoretically - represents a non-trivial didactical research programme.

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